START Selected Topics in Assurance Related Technologies

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Operating Characteristic (OC) Functions and Acceptance Sampling Plans

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Introduction

The Operating Characteristic Function (also known as OC Function) is one of the most useful tools in practical statistical applications. Unfortunately, it is also under utilized and often misunderstood mainly because of confusing information. Some theoreticians think of the OC Function as the result of elaborate calculus-based manipulations, while some practitioners reduce it to a table of values, whose origins are obscure, but whose results are very useful. Missing the strong connections between theory and applications affects the use of the OC Function as the excellent design and analysis working tool that it is.

The objective of this START sheet is to enhance the links between OC Function theory and its practice, by providing an overview of the theoretical background and numerical examples and practical applications. With this, the practicing engineer will hopefully gain an increased awareness of the OC's great potential as a statistical tool, a better understanding of the theory, and be better able to use the OC Function.

First, recall that the OC Function is closely associated with Acceptance Sampling procedures. For it measures the efficiency of a statistical hypothesis test designed to accept or reject a product. We illustrate this use with the following example. Assume we are testing the null hypothesis (H_0) that a batch of widgets is "good" (we accept the batch) versus that it is not (in which case, we reject it). We stipulate that widgets are of good quality when the (unknown) percent defective (PD) in the batch is $p < 0.1$. In the test procedure, a random sample of size $n = 10$ is drawn and the batch is accepted if we find zero ($c = 0$) defectives in it. We define the OC Function for such an acceptance test procedure, conditional on the unknown PD "p," and denote it OC (p), in the following way:

OC (p) = Prob {Batch is accepted, given true, unknown "p"} = Prob {Observing Zero defectives, given PD "p"}

All other things equal, we see that as the PD "p" increases, the probability of finding one defective item in the sample of size $n = 10$ also increases (and the contrary occurs if "p" decreases). If batch items are drawn with replacement and at random, then the number of defectives "x" found in the sample of size $n = 10$, given $p = 0.1$, is distributed as a Binomial with parameters $n = 10$, $p = 0.1$, and is denoted B (x; n, p). The OC Function for such Binomial ($n = 10$, $p = 0.1$) is now written as follows.

$$OC(0.1) = P(x = 0) = B(x = 0; n = 10; p = 0.1) = C^{10}_0\, 0.1^0(1 - 0.1)^{10-0} = 0.9^{10} = 0.349$$

This means that about 34.9% of the times we will find no defects in the sample of $n = 10$ items, when $p = 0.1$. However, if in the same acceptance test above we let PD increase to $p = 0.25$ we see how the OC Function (probability of finding zero defects) decreases to 0.056 or 5.6% of the times:

$$OC(0.25) = P(x = 0) = Bin(x = 0; n = 10; p = 0.25) = C^{10}_0\, 0.25^0(1 - 25)^{10-0} = 0.75^{10} = 0.056$$

In general, for a fixed sample size "n," we can let test parameter "p" vary in the interval (0, 1). We then compute and graph the "OC Function" as p varies. Such graph is called the OC Curve and, for the Binomial test plan above described, is tabulated and shown in Figure 1.

Let the sample size "n" now increase from 10 to 20. However, assume we will still accept the batch if we observe

Row	PD	OC(PD)
1	0.025	0.776330
2	0.050	0.598737
3	0.075	0.458582
4	0.100	0.348678
5	0.125	0.263076
6	0.150	0.196874
7	0.175	0.146063
8	0.200	0.107374
9	0.225	0.078166
10	0.250	0.056314

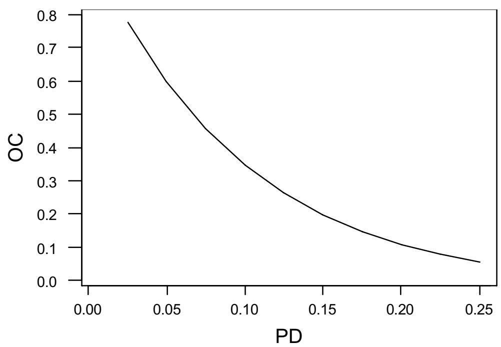

Figure 1. OC for Binomial Test Plan Described in Text

zero defectives (c = 0) in the sample. Then, if the true PD is p = 0.1, OC (p) or the probability of accepting the batch given this larger sample size, decreases as shown in the following equation.

$$OC(0.1) = P(x = 0) = B(x = 0; n = 20; p = 0.1) = C^{20}_0\, 0.1^0(1 - 0.1)^{20-0} = 0.9^{20} = 0.122$$

Finally, we can also modify the acceptance number "c" (maximum number of defectives that can appear in our sample of n = 10, and we still accept the batch). Assume that we increase "c" from Zero to One (i.e., we accept the batch if there are no defectives, or if there is only one defective: $x \leq 1$). Then, the OC Function at p = 0.1, for c = 1 becomes:

$$OC(0.1) = P(x \leq 1) = P\{X = 0\} + P\{x = 1\} = B(x \leq 1; n = 10; p = 0.1) = C^{10}_0\, 0.1^0\, (1 - 0.1)^{10-0} + C^{10}_1\, 0.1^1\, (1 - 0.1)^{10-1} = 0.9^{10} + 10 \times 0.1 \times 0.9^9 = 0.349 + 0.387 = 0.736$$

From the preceding, we see in this relatively simple case (Binomial distribution) that the OC Function depends on the PD (p), the sample size (n) and the acceptance number (c). This triple dependency yields one of the most important uses of the OC Function: deriving Acceptance Sampling Plan tables and "nomographs" to determine the best Plan (n, c), for a sample of size "n" and acceptance number "c", that provides a pre-established "confidence" in our acceptance test results, given the value of our parameter of interest (e.g., "p").

Before closing this section, we need to introduce some more formal definitions of related pairs of concepts that will be used throughout this START sheet:

Null Hypothesis (H_0): assumed status (e.g., the product is of good quality); Alternative Hypothesis (H_1): the negation of the null; the alternative situation.

α probability (also called producer's risk): the probability of deciding that the alternative hypothesis (H_1) is true, when in fact the null (H_0) is true (e.g., risk of rejecting the batch as defective, when it is spec-compliant); β probability (also called the consumer's risk): the probability of deciding that the null hypothesis (H_0) is true, when the alternative (H_1) is true (e.g., the risk of accepting a defective product).

Acceptable Quality Level (AQL): a percent defective that is the base line requirement for the quality of the producer's product. Lot Tolerance Percent Defective (LTPD): a pre-specified high defect level that would be unacceptable to the consumer. (Reference 9)

Producers design acceptance sampling plans with high probability of accepting lots with defect levels less than or equal to the AQL. Consumers prefer sampling plans with a low probability of accepting lots with defects level as high as the LTPD. The best plans are those that, using the OC Function as a design tool, consider both AQL and LTPD levels.

In the rest of this START sheet, we derive and illustrate with practical numerical examples, the OC Functions for the Hypergeometric, Binomial, Poisson, Normal, and Exponential distributions, and their implementation (and their equivalent graphs and tables) to calculate acceptance plans, sample sizes, and confidences that meet our requirements.

Statistical "Confidence" and Plan Parameters: c, p, and n

One of the main uses of the OC Function is to provide the sample size requirement "n" and the statistical "confidence," given the population parameter (p) of interest (Reference 1). For the previous example, a sample size of n = 20 will provide a "confidence" of 87.8% (0.878 = 1-0.122) that the PD is $p \leq 0.1$, when zero defectives (acceptance number c = 0) are found.

We calculate such "confidence" by using the OC Function, as follows. Consider the null hypothesis H_0: "the widgets are acceptable" (because PD is p < 0.1). Then, we state:

$$OC(p) = \text{Prob}\{\text{Accepting } H_0 \text{ if } p < 0.1\} = P\{x \leq c\} = \sum_{x=0}^{c} B(x; n, p)$$

Then: $P\{x > c\} = 1 - P\{x \leq c\} = 1 - OC(p)$ = Prob {Rejecting H_0, given p < 0.1} provides the statistical "confidence" that p < 0.1 when we get "c" defectives or less in the sample of "n" widgets. Such probability $P\{x > c\}$ corresponds to the "p-value" of the hypothesis test mentioned and the sample drawn. To obtain "Confidence," we use the OC Function.

In the example, we obtained that OC (c = 0; p = 0.1, n = 20) = 0.122. Therefore, we can say that the "Confidence" is: 1 - OC (p) = 0.878. Furthermore, we say that we are 87.8% confident that, since we found no defectives in this sample of size n = 20, the true but unknown PD is $p \leq 0.1$.

Analogously, we can pre-establish the "Confidence" = 1 - OC (p) = 0.9 (90%). Then, for arbitrary parameters "c and p," we can find the sample size value "n" that meets such Confidence criteria. If in our example we had required a 90% Confidence that $p \leq 0.1$, from a sample with no defective items (c = 0), then sample size would have been n = 22:

$$OC(0.1) = B(x = 0; n; p = 0.1) = C^n_0\, 0.1^0 (1 - 0.1)^{n-0} = 0.9^n = 0.1 \Rightarrow n \approx 22$$

The reader can consult (<http://rac.alionscience.com/Toolbox/OneShotCalc.htm>) the RAC Binomial Calculator (Reference 2a) and obtain these same results with parameters: p = 0.1, n = 20 and r = 0 (we discuss other RAC calculators in the following sections). There are also "nomographs" that can be used to establish the OC Function parameters of interest, p, c, n, or "confidence," which characterize a sampling plan (see References 3, 4, 5, 6, 7, and 8).

Acceptance Plans that Consider Lot Tolerance Percent Defective (LTPD)

Often, the consumer will not accept a product if the parameter of interest is beyond a pre-specified value (i.e., LTPD), while gladly accepting those with values up to the AQL. In such Acceptance Sampling plans, the producer and consumer risks (α, β) and the OC Function allow us to come up with the adequate sample sizes that fulfill the requirements.

We illustrate the situation for the Binomial. Assume that in the case just discussed, we want to accept the batch (H_0), if the PD is 0.1 or less (AQL). Assume we want to reject the batch, if the PD is greater than 0.15 (LTPD). Also assume that the required errors (α, β) are both, 0.1. Then, to find the sample size "n" that meets such acceptance sampling plan requirements, we need to solve the following system of two OC Function equations:

$$OC(p = 0.1) = P\{\text{Correctly Accepting Good Batch}\} = \sum_{x=0}^{c} B(x; n; p = 0.1) = 1 - \alpha = 0.9$$

$$OC(p = 0.15) = P\{\text{Incorrectly Accepting Bad Batch}\} = \sum_{x=0}^{c} B(x; n; p = 0.15) = \beta = 0.1$$

This system is not linear and is solved using an iterative method. Fix values "n, c" and obtain the corresponding OC Function values for p = 0.1 and 0.15. If they fit the problem requirements, stop. Otherwise, modify values "n, c" and iterate until they do.

However, such an iterative procedure is time-consuming. Hence, we resort to the tables of "Binomial Nomographs" (References 4, 5, and 8). In the margin axes, we enter four values: the two PD, say p_0 and p_1, and their respective, required OC Functions, OC (p_0) = 1 - α and OC (p_1) = β. Then, we draw the lines that joint these margin values. The values of sample size "n" and acceptance number "c" are read from the nomograph values, closer to where the two lines cross. In our present case, the nomograph results show that we require n = 300 data points and an acceptance number c = 36. We check them by solving the above equations using these values of n and c. We obtain OC (0.1) = 0.892 and OC (0.15) = 0.082, close enough to our postulated values of 0.90 and 0.10 (for, n and c being integers, will rarely yield the exact postulated values).

For comparison, if we increase the LTPD p_1, from 0.15 to 0.2, and risk β from 0.1 to 0.2, then the corresponding acceptance sampling plan results become: n = 55 and c = 8. This tells us that, in order to decrease the sample size "n" we need to increase LTPD, the Lot Tolerance Percent Defective (minimum unacceptable value) and to increase the error of accepting a batch, with such unacceptable LTPD, as "good" (i.e., complying with AQL). To make this point clearer, we show in Table 1, the values "n" and "c," for risks α = 0.05 and β = 0.1, and PD values AQL = 0.04, 0.1 and 0.15, and of LTPD = 0.1, 0.15 and 0.2. That is, Table 1 provides the Acceptance Plan (sample size "n" and acceptance number "c") that meets the above criteria with: OC (AQL) = 1 - α = 0.95 and OC (LTPD) = β = 0.1.

Table 1. Plan for Errors α = 0.05 and β = 0.1; where OC (AQL) = 1 - α = 0.95 and OC (LTPD) = β = 0.1

For:	AQL = 4%	AQL = 10%	AQL = 15%
LTPD = 10%	(140, 9)	N/A	N/A
LTPD = 15%	(52, 4)	(355, 44)	N/A
LTPD = 20%	(32, 3)	(125, 18)	(500, 88)

We can see how, as we increase the difference between AQL and LTPD (e.g., at 4% and 20%) a Plan with smaller sample size is required (32, 3) and the contrary occurs as the difference between AQL and LTPD decreases (e.g., at 10% and 15%). This effect is due to the fact that more "information" (n and c) is required when we need to discriminate between two quantities that are close to each other, than when they are far apart. Such discrimination becomes yet more difficult (requiring a larger n, c) if the values of AQL and LTPD are closer to the center of the interval (e.g., 15% and 20%).

The OC Function and the Hypergeometric Distribution

Sometimes the total population (say the batch size N) from which the samples of size "n" are drawn, is finite and (N-n)/N is much smaller than unity. This occurs when "N" is not "large" with respect to "n." If, in addition, we sample without replacement (e.g., as when conducting destructive testing) then the exact distribution of the number "x" of defectives observed in our samples is no longer Binomial, but Hypergeometric.

For example, let the population of interest be an aircraft (AC) fleet of (finite) size N =100 helicopters. Assume that we draw a random sample of size n = 5, without replacement. If there are K =10 (out of the total N) defective AC then, the probability P(x) that we include exactly "x" defective AC in our sample, from among the total K defective AC in the fleet, is given by the Hypergeometric distribution:

$$P(x) = P\{x; N, K, n\} = \frac{C_x^K \, C_{n-x}^{N-K}}{C_n^N}; \text{ where } C_x^K = \frac{K!}{x!(K - x)!}$$

Notice how the probability P(0) of observing zero defective AC (x = 0) in the sample of size "n" (which yields the OC Function) requires the values N, K and the sample size "n", not always available. In the example of 10% defective AC in the fleet of N = 100 the OC Function is: "probability that none of the n = 5 sample AC comes from the K defectives":

$$OC(p) = P(0) = P\{x = 0; N = 100, K = 10, n = 5\} = \frac{C_0^{10} C_{5-0}^{100-10}}{C_5^{100}}$$

$$= \frac{(100 - 10)!/[(5 - 0)! \text{ x } (100 - 10 - (5 - 0))!]}{100!/[5! \text{ x } 95!]} = \frac{90! \text{ x } 5! \text{ x } 95!}{100! \text{ x } 85! \text{ x } 5!}$$

$$= \frac{86 * 87 * 88 * 89 * 90}{96 * 97 * 98 * 99 * 100} \approx 0.584$$

Hence, finding no defective AC in samples of size 5, when N = 100 and K = 10, occurs in about 60% of the cases. If we were to decide (H_0) that the fleet is good to fly, when K < 10, and we would find no defective AC in our sample n = 5, we would be taking a wrong decision (giving the OK to a fleet with K = 10 defectives) 58.4% of the times. However if we increase the sample size to n = 20, then the probability of getting no defective AC in the sample, when there are really K = 10 in the population (and therefore, p = 0.1) is now:

$$OC(p) = P(0) = P\{x = 0; N = 100, K = 10, n = 20\}$$

$$= \frac{90! \text{ x } 20! \text{ x } 80!}{100! \text{ x } 70! \text{ x } 20!} = \frac{71 \text{ x ... x } 80}{91 \text{ x ... x } 100} \approx 0.095$$

By doubling the sample size, the error risk (given by the OC) decreased six times. This example provides yet another important application of OC the Function: establishing the minimal sample size "n" such that, with a pre-specified probability, there is at least one item in the sample drawn, with the characteristic of interest (in the present case, having a defective AC). This is important when we actively want such characteristic to surface, if it exists.

In practice, we first establish the probability, P(x), of finding "x" items with the given characteristic. Then, using the OC Function, we obtain the required sample size "n." Since for this we need to know the number K of defective items in the population (or an initial estimate of it), we consider several values for K. Then we see what sample sizes "n" yield the pre-specified probability. The size N of the population is fixed, hence the ratio K/N of defective items moves in the range (0, 1) as K increases. The result is the OC Function (probability of assuming fleet is good, given K) which allows us to find a suitable sample size "n" for the population total N and number of items defectives K (or PD "p").

When the population size (N) is large with respect to the sample size (n) and n/N ≈ 0, as in our current example, we can approximate the Hypergeometric by the Binomial distribution. Letting p = K/N, we obtain the approximate results. Using the RAC Binomial Calculator (2a) we can also approximate the exact Hypergeometric distribution in the previous example, by letting the sample size n = 5 and p = 0.1.

$$P(0) = P\{x = 0; N = 100, K = 10, n = 5\} \approx B(x = 0; n = 5, p = 0.1) = C^n_x p^x (1 - p)^{n-x} = C^5_0 0.1^0 (1 - 0.1)^{5-0} = 0.9^5 = 0.59$$

The exact statistical model remains the Hypergeometric distribution. However, because the population size N is large and that n/N is small (say, less than 0.1), the Binomial probabilities are reasonably close to the Hypergeometric results.

The OC Function and the Poisson Distribution

If N is very large with respect to K, or the PD p is very small, the problem can then be modeled with the Poisson distribution. In such case Poisson's parameter is $\lambda = n \times p$. For our above example $\lambda = 5 \times 0.1 = 0.5$. Hence, the corresponding (approximate) probability of finding no defective items (c = 0) and of accepting (H_0) that the fleet is good to fly is:

$$P(0) = P\{x = 0; N = 100, K = 10, n = 5\} \approx B(x = 0; n = 5, p = 0.1)$$

$$\approx \text{Poisson}(x = 0; \lambda = np = 0.5) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.5} \, 0.5^0}{0!}$$

$$= e^{-0.5} = 0.603$$

Here, we also need to have prior knowledge of Poisson parameter λ, which implicitly requires the knowledge of "p" and of sample size "n." As before, one obtains the OC Function for different values of "p" and determines, for the required "confidence," the adequate sample size "n" that fulfills such problem requirements. We can use the RAC Poisson Calculator (Reference 2b) to assist us in these calculations.

We illustrate how the OC Function is obtained for the Poisson distribution. As before, assume that the fleet is good to fly (H_0) when "c" or less defective AC are found in the sample of size "n", and "p" is the true fraction defective. The acceptance number "c" is again the maximum number of defective AC that we can observe in the sample, and still decide that the fleet is "good to fly." Then, the general expression for OC ($\lambda = n \times p$) is:

$$OC(\lambda = n \text{ x } p) = P\{X \leq c\}$$
$$= P_p\{X = 0\} + P_p\{X = 1\} + ... + P_p\{X = c\}$$

Letting "p" run from 0 to 1, we obtain λ (= n × p) and graph OC (λ), for c = 0 and n = 5 as shown in Figure 2.

Row	Lambda	OC (Lambda)
1	0.01	0.99
2	0.05	0.95
3	0.20	0.83
4	0.45	0.64
5	0.70	0.50
6	0.95	0.39
7	1.20	0.30
8	1.45	0.23
9	1.70	0.18
10	1.95	0.14
11	2.20	0.11

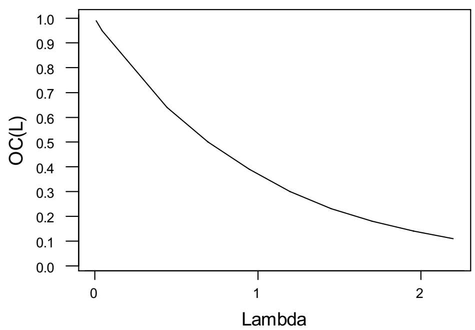

Figure 2. The OC for c = 0 (Poisson Distribution)

In Table 2, we compare the three OC Function procedures implemented, for a population of N = 100 AC, for K = 10 AC defectives and sample size n = 5, using the Hypergeometric, Binomial and Poisson. The approximation case (p = n/N = 0.1) is borderline for all three.

Table 2. Comparing the OC Function for the Hypergeometric, Binomial, and Poisson Distributions

Distribution	$OC_{p=0.1}(X \leq c = 0)$
Hypergeometric	0.584
Binomial	0.591
Poisson	0.603

Assuring a Pre-specified Number of Items of Interest in the Sample

We now discuss how Poisson, Hypergeometric and Binomial OC Function nomographs and formulas are used to calculate the sample size required so that, with a pre-establish probability, at least one sample element exhibits some characteristics of interest that appears in the population, with true (but unknown) probability of occurrence "p".

To set this problem in perspective, assume there is a characteristic (say a particular type of helicopter blade failure) that occurs with probability p = 0.02, in the sub-population of AC failures (e.g., 2% of the times a failure occurs, it is a blade failure). We want to draw a sample of AC failures, of size "n", large enough so that it includes at least "c" blade failures, with probability say, OC (p) = 0.9, i.e., 90% of the time.

For large populations "N" we can approximate the exact Hypergeometric distribution by the Binomial. We can then calculate, via the OC Function, how many failures "n" we need to obtain, to see at least "c" of the "special" types (e.g., blade failures) among them:

$$OC(p) = P\{X \leq c\} = \sum_{x=0}^{c} C_x^c \, p^x \, (1-p)^{c-x}$$

We start by letting n = 100, p = 0.02 and c = 0. Then, we obtain from the Binomial (n, p) tables:

$$OC(p) = P\{X = 0\} = (1 - 0.02)^{100} = 0.1326$$

This means that, in a sample of n = 100 failures, we will observe c = 0 "special" failures with probability 0.13. Hence, we will have at least one failure of this particular type, with probability P{X > 0} = 1 - 0.133 = 0.867 (i.e., 86.7% of the time). If we increase the sample size, for example to n = 110, we obtain a higher probability: OC (p) = $(1- 0.02)^{110}$ = 0.108. This yields a confidence of 1 - 0.108 ≈ 0.9, of seeing at least one failure in the sample, 90% of the time.

Assume we need more than one failure of this "type," say three or more: then c = 2. From the Binomial tables, for p = 0.02 and n = 280, we obtain the OC Function for {X ≤ 2}:

$$OC(p) = P\{X \leq 2\} = P\{X = 0\} + P\{X = 1\} + P\{X = 2\}$$
$$= 0.004 + 0.0199 + 0.0568 = 0.081$$

With a sample of n = 280 we will observe up to c = 2 failures, with probability 0.081. Hence, with confidence P{X > 2} = 1 - OC (p) = 1 - 0.081 = 0.919 we will observe more than two failures of this "special" type (i.e., 91.9% of the time). The RAC Binomial Calculator (Reference 2a) can be used to obtain similar results, by inputting values r = 2, p = 0.02 and n = 287.

More over, when sample sizes "n" are large and the PD "p" is small, we can approximate the Binomial with parameters n = 280, p = 0.02 by the Poisson with parameter λ = n × p = 280 × 0.02 = 5.6, obtaining similar results:

$$OC_{c=2}(p) = P\{X \leq c = 2\} = \sum_{x=0,2} \frac{e^{-\lambda}\lambda^x}{x!}$$
$$= 0.004 + 0.021 + 0.058 = 0.083$$

In many cases, such values can also be obtained from nomographs (References 3 to 8) that are based on the principles and formulas described in this START sheet.

OC Functions for the Normal Distribution

We overviewed the problem of deriving OC Functions for three discrete distributions that arise when sampling "attributes": Hypergeometric, Binomial, and Poisson. An analogous situation arises when sampling from continuous distributions. We will discuss now the problem of obtaining the OC Function for variables distributed Normal (e.g., Time to Repair, denoted TTR) via a practical example.

Assume a device TTR is Normal, with known variance (say, $\sigma^2 = 25$) and unknown mean (μ) and that (H_0) the device will be used in our system, if its mean (MTTR) μ is less than ten hours ($\mu \leq 10$). In this setting, the OC Function is defined in terms of sample average $\bar{x}_n$ obtained from the sample of "n" TTR, instead of in terms of number of failures "x":

OC (μ) = Prob {The Device is accepted for use, given the true, unknown MTTR "μ"}

Hence, we incorporate the device for use in our system (H_0), if mean $\bar{x}_n$ is below some pre-specified thresh-hold (γ) defined for this purpose. Now, Average $\bar{x}_n$ and Acceptance Number "c," play the same role as "x" and "γ" played, in the discrete case. Both also depend on sample size "n." That is, we will accept the device if its sample MTTR $\bar{x}_n$ is less than or equal to constant "γ" for say, n = 9 TTRs. The OC Function is:

$$OC(\mu) = \text{Prob}\{\text{Accepting } H_0 \text{ Given } \mu\} = \text{Prob}\{\bar{x} \leq \gamma | \mu\}$$

Value γ is chosen so that the probability of rejecting the device 1 - P { $\bar{x}_n \leq \gamma|\mu$), when the true MTTR $\mu = 10$, is the pre-specified error probability (say 5%, or $\alpha = 0.05$). Such percentile value, denoted $z_{1-\alpha}$, is determined from the Normal tables, by calculating:

$$\text{Prob}\{\bar{x} \leq \gamma | \mu\} = \text{Normal}\left(\frac{\gamma - \mu}{\sigma/\sqrt{n}}\right)$$

$$= \text{Normal}\left(\frac{\gamma - 10}{5/\sqrt{9}}\right) = \text{Normal}(z_{1-\alpha}) = 1 - \alpha = 0.95$$

Since the Normal Standard percentile for $1 - \alpha = 0.95$ is $z_{0.95} = 1.65$, then $\frac{\gamma - \mu}{\sigma/\sqrt{n}} = z_{1-\alpha}$ and the thresh-hold becomes:

$$\gamma = \mu + \frac{\sigma}{\sqrt{n}} z_{1-0.05} = 10 + \frac{5}{\sqrt{9}} \times 1.65 = 12.75$$

The OC Function for the sampling plan, defined by $\alpha = 0.05$, $\sigma = 5$, $\gamma = 12.75$, and n = 9 is given by the following equation.

$$\text{Normal}\left(\frac{12.75 - \mu}{5/\sqrt{9}}\right) = \text{Normal}\left(\frac{12.75 - \mu}{1.67}\right); \text{for } \mu > 0$$

As before, the OC Function depends on "n" and is used to determine the sampling plan requirements. For example, the OC table in Figure 3 shows that, when n = 9 and MTTR $\mu = 12$, we will erroneously incorporate the device 67% of the times. If the consumer feels that such MTTR value is unacceptable (LTPD) and wants such error probability to be at most 10% ($\beta = 0.1$) then we will need to increase the sample size. We find an adequate "n" iteratively, by recalculating the size (n), using the Normal 10th Percentile $z_{0.1} = -1.28$:

$$\text{Normal}\left(\frac{12.75 - 12}{5/\sqrt{n}}\right) = \text{Normal}(z_{0.1}) = (-1.28) = 0.1$$

$$\Rightarrow z_{0.1} = -1.28 = \frac{12.75 - 12}{5/\sqrt{n}} \Rightarrow n = \frac{-1.28^2 \times 5^2}{(12.75 - 12)^2} = 72.82$$

Row	Mean U	Prob(U)
1	8	0.997775
2	9	0.987632
3	10	0.950191
4	11	0.852659
5	12	0.673321
6	13	0.440500
7	14	0.227078
8	15	0.088941
9	16	0.025821
10	17	0.005465

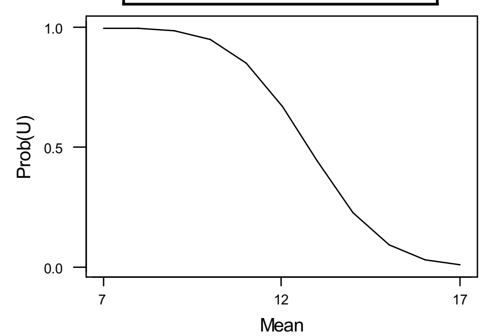

Figure 3. OC for the Normal, $\alpha = 0.05$, $\sigma = 5$, $\gamma = 12.75$ and n = 9

For this "new" value of n = 72, we need to recalculate: $\gamma = \mu + \sigma/\sqrt{n}\, z_{1-0.05} = 10 + 5/\sqrt{72.8} \times 1.65 = 10.96$. We then assess if such (n, γ) meet the Plan requirement: Normal {(10.96-12)/(5√72)} = $\beta = 0.1$. We repeat until this result is achieved. After several iterations we get a Plan with size n = 54, which will incorrectly accept the device only 10% of the times, when the true MTTR is 12 hours, as required, as shown in Figure 4.

Row	Mean	OC(Mean)
1	9	0.999083
2	10	0.950124
3	11	0.569996
4	12	0.097948
5	13	0.002863

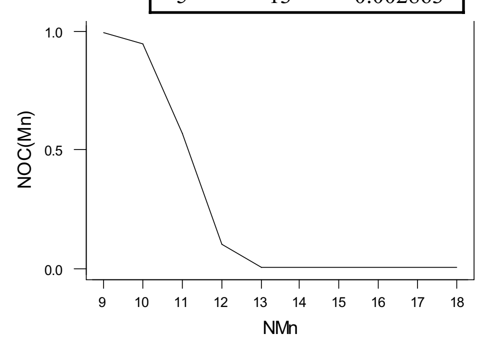

Figure 4. OC for the Normal n=54: OC(10)=0.95; OC(12)=0.1

OC Functions for the Exponential Distribution

The OC Function for the Exponential case is obtained in a similar way. Hence, we will illustrate it using the same example above. Assume that our device Time to Repair (TTR) is now distributed Exponential, with unknown (MTTR) mean θ. As before, assume (H_0) that the device will be used in the system if its MTTR is less than ten hours ($\theta \leq 10$). The OC Function is now defined in terms of the sample total time (T) on test:

$$T = \sum_{i=1}^{n} TTR_i$$

The acceptance approach becomes "accept the devise for use in the system, if T is below a pre-specified thresh-hold (γ) that, as before, depends on the sample size "n". Such "event" is denoted as $\{T \leq \gamma\}$ and its probability is:

OC (θ) = Prob {The Device is accepted for use, given the true, unknown MTTR "θ"}

$$OC(\theta) = \text{Prob}\{\text{Accepting } H_0 \text{ Given } \theta\} = \text{Prob}\{T \leq \gamma|\theta\}$$

For the Normal case, we searched the percentiles in the Normal tables. Now, we search the Chi-Square table. The underlying distribution of the TTR is Exponential. Hence, the distribution of the statistic "twice the Total Test Time, divided by the Mean" (2*T/θ) is distributed as a Chi-Square with 2n Degrees of Freedom (DF). For a pre-specified error probability (α) of rejecting H_0 erroneously, we obtain the α-percentile ($X^2_{\alpha,2n}$) of the Chi-Square, with DF = 2n. Constant γ is chosen so that the probability of rejecting the device 1 - P($T \leq \gamma$) when the true MTTR θ = 10 and n = 9, meets the pre-specified error α = 0.05:

$$OC(\theta) = \text{Chi-Square}\{2 \text{ x } T/\theta \leq 2 \text{ x } \gamma/\theta\} = 1 - \alpha$$
$$= 0.95 \Rightarrow 2 \text{ x } \gamma/\theta = X^2_{0.95,18} = 28.87$$

The Chi-Square percentile for DF = 2n and 1 - α = 0.95, is $X^2_{\alpha,2n}$ = 28.87. The value of γ is:

$$\gamma = X^2_{\alpha,2n} \text{ x } \theta/2 = 28.87 \text{ x } 10/2 = 144.35$$

The OC Function for this plan, defined by α = 0.05, γ = 144.35, and n = 9 is:

$$\text{Chi-Square}\left(\frac{2 \text{ x } \gamma}{\theta}\right) = \text{Chi-Square}\left(\frac{2 \text{ x } 144.35}{\theta}\right); \text{ for } \theta > 0$$

The OC Curve for this Exponential Case is, therefore, as shown in Figure 5.

Row	Mean	OC(Mean)
1	8	0.993127
2	9	0.978480
3	10	0.950018
4	11	0.905736
5	12	0.846888
6	13	0.777120
7	14	0.701073
8	15	0.623228
9	16	0.547236
10	17	0.475689
11	18	0.410173

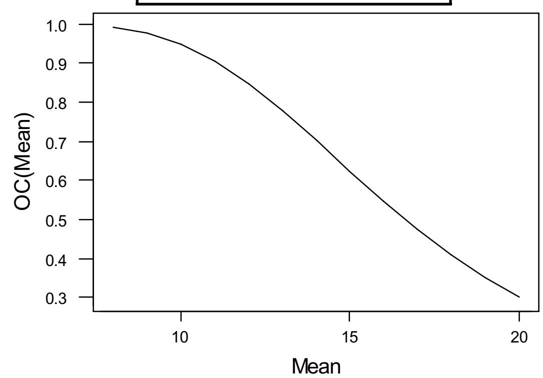

Figure 5. OC Curve for the Exponential Case

Under such Plan, the probability of accepting the device when the true MTTR = 12 hours, is 0.846. Assume such value were unacceptable for the consumer (LTPD) and we require that its probability "β" to be, say at most, 0.1. Then we must, as before, increase sample size n. To find the adequate "n" for this LTPD value, we use the 90th Percentile 10.86 of the Chi-Square ($X_{\beta=0.1,DF=2n}$ = 10.86) and follow the same iterative procedure, indicated in the case above for a Normal OC Function, until we meet the required (α, β) conditions.

Summary and Conclusions

We have seen how the OC Function allows us to obtain acceptance sampling plans that meet both, the producer's and consumer's risks (α, β). The OC also helps us estimate the adequate sample sizes "n" required to obtain, with high confidence, one or more failures in our experiments. These OC Function properties allow us to control the quality of our incoming products, estimate certain parameters of interest such as the required sample sizes, which contribute to the design of better and more efficient experiments.

The implementation of the OC Function has been overviewed for several of the most important discrete and continuous distributions. Implementation for other distributions can be done following the same principles described here. References 3 to 9 deal with industrial statistics and treat further topics of this problem. They include the use of OC Function tables that calculate sample sizes and help avoid the iterative procedures.

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Dr. Jorge Luis Romeu has over thirty years of statistical and operations research experience in consulting, research, and teaching. He is a Certified Reliability Engineer (CRE). He was a consultant for the petrochemical, construction, and agricultural industries. Dr. Romeu has also worked in statistical and simulation modeling and in data analysis of software and hardware reliability, software engineering and ecological problems.

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